REMEMBER THE GOAL: FIND THE ZEROES / X- INTERCEPTS OF POLYNOMIALS.

Theorem <u>Factor theorem</u>

Let $P(x)$ be a polynomial. If $x-a$ is a factor of $P(x)$ then $P(a)=0$. Conversely, if $P(a)=0$, then $x-a$ is a factor.

Proof. ($\Rightarrow$) Assume that $x-a$ is a factor of $P(x)$.

Then $P(x) = (x-a)\cdot \underline{\text{something}}$

If we plug $x=a$, we get

$$P(a) = (a-a)\cdot \underline{\text{something}} = 0.$$

Thus $P(a)=0$.

($\Leftarrow$) Assume $P(a)=0$.

By division algorithm we know that if we divide $P(x)$ by $x-a$ we get

$$P(x) = (x-a)Q(x) + R(x) \quad \text{———} \quad (*)$$

where degree $R(x) <$ degree of $x-a = 1$. So $R(x)$ must be a constant. Let remainder $= R$.

Plug in $x=a$ into $(*)$. Then

$$P(a) = (a-a)Q(a) + R$$

$$\Rightarrow \quad 0 = R \qquad \text{(By assumption)}$$

Thus, remainder is zero. This means $x-a$ divides $P(x)$. $\square$

Example Determine whether $x+2$ is a factor of $P(x) = x^3 - 2x^2 - 5x + 6$. If so, factor $P(x)$ completely.

Soln. Note $x+2 = x-(-2)$

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -5 & 6 \\ & & -2 & 8 & -6 \\ \hline & 1 & -4 & 3 & \boxed{0} \end{array}$$

Since remainder is 0, $x-(-2) = x+2$ divides $P(x)$, i.e. $x+2$ is a factor of $P(x)$. We have

$$P(x) = (x+2)(x^2 - 4x + 3)$$
$$= (x+2)(x-3)(x-1)$$

Exercise Determine whether $x-1$ is a factor of $P(x) = x^3 - 4x^2 - 7x + 10$. If so factor $P(x)$ completely.

Example Determine whether $x-3$ and $x+2$ are factors of $P(x) = x^4 - 13x^2 + 36$. If so factor $P(x)$ completely.

Sol.^{n} Let's see if $x-3$ is a factor.

$$\begin{array}{r|rrrrr} 3 & 1 & 0 & -13 & 0 & 36 \\ & & 3 & 9 & -12 & -36 \\ \hline & 1 & 3 & -4 & -12 & \boxed{0} \end{array}$$

so $P(x) = (x-3)(x^3 + 3x^2 - 4x - 12)$ $(*)$

Now $x+2$ does not divide $x-3$. So if it does divide $P(x)$, it must divide $x^3 + 3x^2 - 4x - 12$.

Let's see if $x+2$ is a factor.

Note $x+2 = x-(-2)$

$$\begin{array}{r|rrrr} -2 & 1 & 3 & -4 & -12 \\ & & -2 & -2 & 12 \\ \hline & 1 & 1 & -6 & \boxed{0} \end{array}$$

Thus,

$$x^3+3x^2-4x-12 = (x+2)(x^2+x-6). \quad (**)$$

Therefore, from (*) and (**) we have

$$P(x) = (x-3)(x+2)(x^2+x-6)$$

Now let's factor x^2+x-6

$$= (x+3)(x-2)$$

$$\begin{array}{r|l} 2 & 6 \\ 3 & 3 \\ & 1 \end{array}$$

Thus we have,

$$P(x) = (x-3)(x+2)(x+3)(x-2)$$

Exercise: Determine whether $x-3$ and $x+2$ are factors of $P(x) = x^4 - x^3 - 7x^2 + x + 6$. If so, factor $P(x)$ completely.

Remark:

For a polynomial of degree n, there are at most n real zeros (counted with multiplicity). In sec. 2.5 we will see that there will be exactly n complex zeros. The real ones will be complemented by complex zeros.

Thm. <u>Rational Zero Theorem</u>

If a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_2x^2 + a_1x + a_0$ has integer coefficients, then every rational zero of $P(x)$ has the form

$$\text{Rational zero} = \frac{\text{factor of } a_0}{\text{factor of } a_n}$$

[<u>Note</u>: This does not guarantee that all numbers of the above form are zeros.]

Not all real numbers are rationals. For example, Greeks already knew that $\sqrt{2}$ is not rational.

<u>Example</u>. Determine possible rational zeros for the polynomial $P(x) = x^4 - x^3 - 5x^2 - x - 6$.

<u>Solun</u>. We have $a_0 = -6$ Factors: $\pm 1, \pm 2, \pm 3, \pm 6$

$a_n = 1$ Factors ± 1.

The possible rational zeros are

$$\frac{\pm 1}{\pm 1}, \quad \frac{\pm 2}{\pm 1}, \quad \frac{\pm 3}{\pm 1}, \quad \frac{\pm 6}{\pm 1}.$$

So the possible rational zeros are

$1, -1, \quad 2, -2, \quad 3, -3, \quad 6, -6.$

Now you have two options to test whether the above are zeros.

<u>Option 1</u>

find $P(a)$

If $P(a) = 0$ then it is a zero.

<u>Option 2</u>

Synthetic division. If $x-a$ divides $P(x)$, then a is a zero

Option 1 $P(1) = 1^4 - 1^3 - 5\cdot 1^2 - 1 - 6 = 12$

Option 1 $P(-1) = (-1)^4 - (-1)^3 - 5(-1)^2 - (-1) - 6 = -8.$

Option 2 2 is zero?

$$\begin{array}{r|rrrrr} 2 & 1 & -1 & -5 & -1 & -6 \\ & & 2 & 2 & -6 & -14 \\ \hline & 1 & 1 & -3 & -7 & \boxed{-20} \end{array}$$

No.

Option 2 -2 is zero?

$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -5 & -1 & -6 \\ & & -2 & 6 & -2 & -6 \\ \hline & 1 & -3 & 1 & -3 & \boxed{0} \end{array}$$

Thus, -2 is a zero, i.e. $(x+2)$ is a factor

Hence,

$$P(x) = (x+2)(x^3 - 3x^2 + x - 3). \qquad (*)$$

Option 2 3 is zero?

$$\begin{array}{r|rrrr} 3 & 1 & -3 & 1 & -3 \\ & & 3 & 0 & 3 \\ \hline & 1 & 0. & 1. & \boxed{0} \end{array}$$

Thus, $x^3 - 3x^2 + x - 3 = (x-3)(x^2 + x) \quad (**)$

By $(*)$ and $(**)$

$$P(x) = (x+2)(x-3)(x^2+1).$$

Note x^2+1 cannot be factored.

Descarte's Rule of Signs

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$.

(i) The number of positive zeros is either equal to the number of sign changes or less than that by an even number

(ii) The number of odd zeros is either equal to the number of sign changes of $P(-x)$ or less than that by an even number.

Ex. Determine the possible combinations of real zeros for

$$P(x) = x^4 - 2x^3 + 2x - 2$$

Sign chang   Sign change   Sign Change

By Descarte's Rule of signs, there are 3 or 1 positive real zero.

$$P(-x) = (-x)^4 - 2(-x)^3 + 2(-x) - 2$$
$$= x^4 + 2x^3 - 2x - 2$$

Sign change

By Descarte's Rule of signs, there is 1 negative real zero.

Since $P(x)$ is a degree 4 polynomial, there are "at most" 4 real zeros. One is a negative real number and there is at least 1 positive real zero.

Exercise. Determine the possible combination of zeros for
$$P(x) = x^4 + 2x^3 + x^2 + 8x - 12.$$

Ques. Why is Descartes Rule of Signs useful?

Ans. It helps us identify the possible combination of positive and negative real zeros. So we can discard a number of possibilities while checking for rational roots making life easier.

Example. Write $P(x) = x^5 + 2x^4 - x - 2$ as a product of linear and/or irreducible quadratic factors.

Solun. Step 1 (Descartes)

$P(x)$ has 1 sign change.

So there is one positive real zero.

$$P(-x) = -x^5 + 2x^4 + x - 2$$

There are 2 sign changes.

So there are either 2 negative real zeros or 0 negative zeros.

Step 2 (Rational Root)

$a_0 = -2$ Factors $\pm 1, \pm 2$

$a_n = 1$ Factors ± 1

The possible rational zeros are $\pm 1, \pm 2$.

From step 1 there is only one positive real zero.

Testing:

$$\begin{array}{r|rrrrrr} 1 & 1 & 2 & 0 & 0 & -1 & -2 \\ & & 1 & 3 & 3 & 3 & 2 \\ \hline & 1 & 3 & 3 & 3 & 2 & \boxed{0} \end{array}$$

1 is a zero. No need to check for 2.

Now there is either 2 or 0 negative real zeros.

Testing

$$\begin{array}{r|rrrrr} -1 & 1 & 3 & 3 & 3 & 2 \\ & & -1 & -2 & -1 & -2 \\ \hline & 1 & 2 & 1 & 2 & \boxed{0} \end{array}$$

-1 is a zero.

Testing for another one.

$$\begin{array}{r|rrrr} -2 & 1 & 2 & 1 & 2 \\ & & -2 & 0 & -2 \\ \hline & \underbrace{1 \quad 0 \quad 1}_{x^2+1} & & & \boxed{0} \end{array}$$

3 Zeros are $1, -1$ and -2.

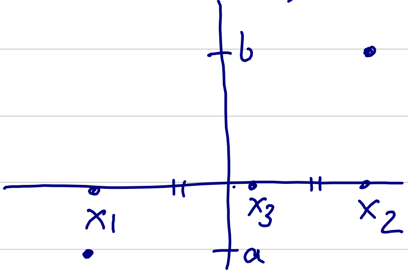
$$P(x) = (x-1)(x+1)(x+2)(x^2+1)$$

Exercise: Write $P(x) = x^5 - 2x^4 + x^3 - 2x^2 - 2x + 4$ as a product of linear and/or irreducible quadratic factors.

Exercise

Write $P(x) = 3x^4 - 5x^3 - 17x^2 + 13x + 6$ as a product of linear and/or irreducible quadratic factors.

Bisection Method (Newton)



Let's say $f(x)$ is a continuous function. Say that the value at x_1 and x_2 have opposite signs. By the intermediate value theorem, we know that there is a zero somewhere between x_1 and x_2. Assume that $f(x_1) = a < 0$ & $f(x_2) = b > 0$.

We will take the midpoint $x_3 = \frac{x_1 + x_2}{2}$ and test the value of the function f at x_3. If $f(x_3) < 0$ then we know that there is a zero between x_3 and x_2. If $f(x_3) > 0$, then we know that there is a zero between x_1 and x_3. If $f(x_3) = 0$ then we are done.

Otherwise we continue taking midpoints and follow the above steps.

[Newton did this when they were no calculators around.]

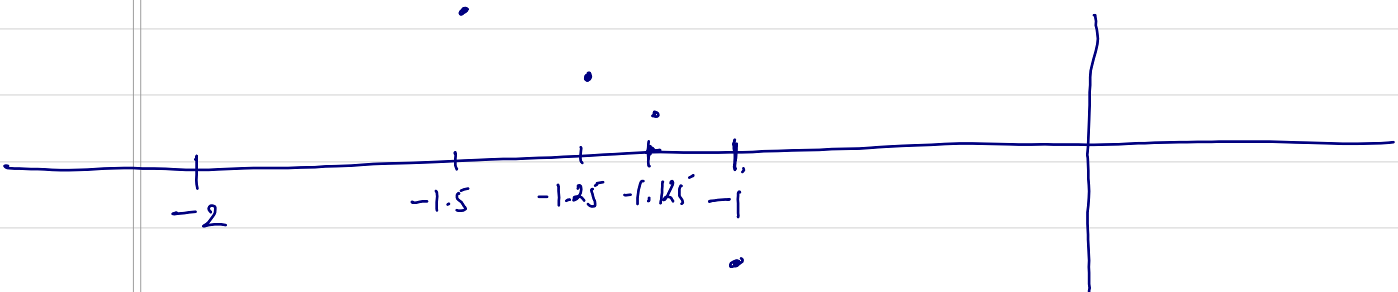
Example Use the IVP and bisection method to approximate the real zero in the indicated interval. Approximate to one decimal place

$$f(x) = x^6 + 2x^4 - 3x^2 - x - 2 \quad \text{in the interval } [-2,-1].$$

Solution.

$$f(-2) = (-2)^6 + 2(-2)^4 - 3(-2)^2 - 2 - 2 = 84$$

$$f(-1) = (-1)^6 + 2(-1)^4 - 3(-1)^2 - (-1) - 2 = -1.$$



Since $f(-2)$ and $f(-1)$ have opposite signs, by the intermediate value theorem there is a zero between -2 and -1. The midpoint of $[-2,-1]$ is

$$c = \frac{-1+(-2)}{2}$$

$$= \frac{-3}{2}$$

$$= -1.5$$

Then $f(-1.5) = 14.2656 > 0$.

Since $f(-1.5)$ and $f(-1)$ have opposite signs, by the IVP there is a zero in $[-1.5,-1]$. The midpoint of $[-1.5,-1]$ is $c = \dfrac{-1.5+(-1)}{2}$

$$= -1.25$$

we have

$$f(-1.25) = (-1.25)^6 + 2(-1.25)^4 - 3(-1.25)^2 - (-1.25) - 1$$
$$= 3.26001$$

Hence, by the IVP there is a zero between -1.25 and -1.

The midpoint of $[-1.25, -1]$ is

$$\frac{-1.25 + (-1)}{2}$$
$$= \frac{-1.25 - 1}{2}$$
$$= -1.125$$

we have

$$f(-1.125) = 0.559025$$

Thus by the IVP there is zero in $[-1.125, -1]$.

Note that we want the zero approximate upto 1 decimal place.

The distance from the midpoint of $[-1.125, -1]$ to its endpoints is less than 0.1. Thus, the distance from the midpoint $= \frac{-1.125 + (-1)}{2}$

$$= -1.0625$$

will be less that 0.1. This is because the zero will be closer to the midpoint than the endpoints.

Thus, required number is -1.0625.